EE 553 Homework 4
Solution

Fall 2023

Problem 1 (25 points)

A lossless power system has a demand of 700MW, which is met with 4 generators with the following linear cost curves and generation limits.

$$
\begin{aligned}
C_1(P_{G1}) &= 500 + 15P_{G1} \text{ \$/hr} \qquad & P_{G1} &\in [0, 300] \\
C_2(P_{G2}) &= 300 + 20P_{G2} \text{ \$/hr} \qquad & P_{G2} &\in [0, 150] \\
C_3(P_{G3}) &= 200 + 25P_{G3} \text{ \$/hr} \qquad & P_{G3} &\in [0, +\infty) \\
C_4(P_{G4}) &= 200 + 23P_{G4} \text{ \$/hr} \qquad & P_{G4} &\in [0, +\infty)
\end{aligned}
$$

A transmission line presents the following operative constraint:

$$0.5P_{G1} + 0.3P_{G2} - 0.2P_{G3} \leq 100$$

Write a Simplex method to solve this problem. Report the program code and the tableau at each iteration. What are the optimal solution and the optimal value of the objective function?

Solution. The MATLAB code for the Simplex method is

```
    tbl = [ 1     1     1     1   0   0   0     700
            1     0     0     0   1   0   0     300
            0     1     0     0   0   1   0     150
            0.5   0.3  -0.2   0   0   0   1     100
           -8    -3     2     0   0   0   0  -17300];
    ibv = [4 5 6 7];          % indices of basic variables
    inbv = [1 2 3];           % indices of nonbasic variables

    icost = size(tbl,1);      % row of reduced cost in the tableau
    iA = 1:icost-1;           % row indices of data matrix in the tableau
    ib = size(tbl,2);         % col of b in the tableau

    ienter = 1;               % index of init. NBV (in inbv) to enter the basis

    iter = 0;
    while ~isempty(ienter)
        % find the basic variable to exit the basis
        eps = tbl(iA, ib) ./ tbl(iA, inbv(ienter));
        eps(eps < 0) = inf; % skip rows with negative coeff.
        [v, iexit] = min(eps);

        % update the tableau by Gaussian elimination
        tbl(iexit, :) = tbl(iexit, :) / tbl(iexit, inbv(ienter));
        for i = 1:icost
            if i ~= iexit && tbl(i, inbv(ienter)) ~= 0
                tbl(i, :) = tbl(i, :) - tbl(i, inbv(ienter)) * tbl(iexit, :);
```

```
            end
        end

        % update the basic/nonbasic variable list
        tmp = inbv(ienter);
        inbv(ienter) = ibv(iexit);
        ibv(iexit) = tmp;

        % find the nonbasic variable to enter the basis
        [rc, ienter] = min(tbl(icost, inbv));
        if rc >= -1e-8
            ienter = [];
        end
        iter = iter + 1;

        fprintf('Iteration %i\n============\n', iter);
        fprintf('Tableau:\n')
        fprintf(1,'%7.3f %7.3f %7.3f %7.3f %7.3f %7.3f %7.3f %9.2f\n',tbl.');
        fprintf('Indices of basic variables: %i %i %i %i\n',ibv);
        fprintf('\n');
    end
```

The output is

```
Iteration 1
============
Tableau:
  0.000   0.400   1.400   1.000   0.000   0.000  -2.000    500.00
  0.000  -0.600   0.400   0.000   1.000   0.000  -2.000    100.00
  0.000   1.000   0.000   0.000   0.000   1.000   0.000    150.00
  1.000   0.600  -0.400   0.000   0.000   0.000   2.000    200.00
  0.000   1.800  -1.200   0.000   0.000   0.000  16.000 -15700.00
Indices of basic variables: 4 5 6 1

Iteration 2
============
Tableau:
  0.000   2.500   0.000   1.000  -3.500   0.000   5.000    150.00
  0.000  -1.500   1.000   0.000   2.500   0.000  -5.000    250.00
  0.000   1.000   0.000   0.000   0.000   1.000   0.000    150.00
  1.000   0.000   0.000   0.000   1.000   0.000   0.000    300.00
  0.000  -0.000   0.000   0.000   3.000   0.000  10.000 -15400.00
Indices of basic variables: 4 3 6 1
```

The optimal solution is $\underline{P_G^* = [300, 0, 250, 150] \text{ MW}}$, and the optimal objective value is $\underline{15400 \text{ \$/hr}}$. ■

Problem 2 (25 points)

Consider the 3-bus power system in Fig. 1 where the line reactance are $x = 0.2$ p.u. and thermal limits are 80 MW for all lines. The generators have the following operating costs:

$$C_1(P_{G1}) = 8P_{G1} \text{ \$/hr} \qquad P_{G1} \in [0, 100]$$
$$C_2(P_{G2}) = 10P_{G2} \text{ \$/hr} \qquad P_{G2} \in [0, 100]$$
$$C_3(P_{G3}) = 15P_{G3} \text{ \$/hr} \qquad P_{G3} \in [0, 100]$$

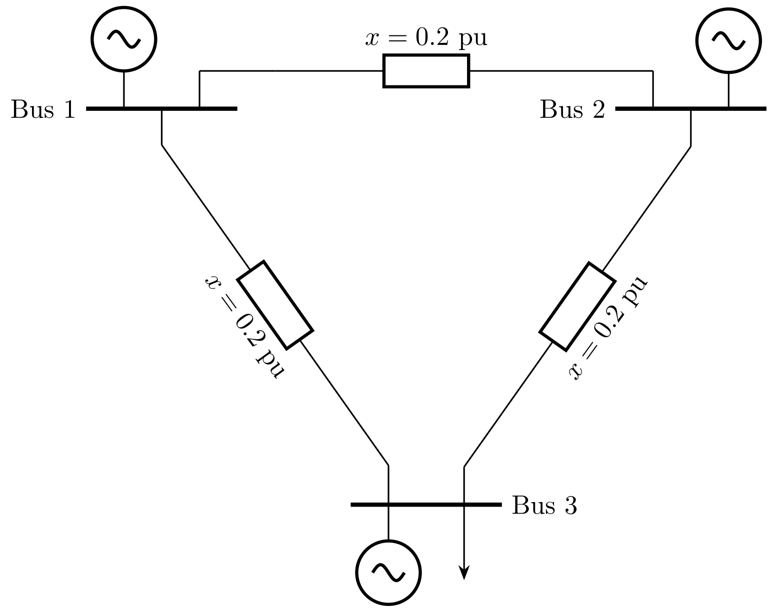

Figure 1: Three-bus system in Problem 2.

a) Develop a plot of the LMP at bus 3 as a function of the load at bus 3 for a range $[0, 250]$ MW, considering transmission line and generator limits in normal operation (no contingencies). Show all steps.

   Solution. It is obvious that when no constraints are imposed, the most economic option is to dispatch entirely from generator 1. So when the load is less than 100 MW, the power is supplied entirely from generator 1 and the LMP at bus 3 is \$8. When the load is between 100 and 140, the incremental generation is supplied by generator 2 and the marginal cost is therefore \$10. When $P_{G1} = 100$ and $P_{G2} = 40$, $P_{13} = 2P_{G1}/3 + P_{G2}/3 = 80$ is at the limit. Further increase of P_{G2} needs to be accompanied by corresponding decrease of P_{G1} to ensure the line limit is not violated. Specifically, to make sure $P_{13} = 2P_{G1}/3 + P_{G2}/3$ stays at 80, to increase the load at bus 3 by every 1 MW, generator 2 generates 2 MW more and generator 1 generates 1 MW less. So the marginal cost is $2 \times 10 - 8 = \$12$. When P_{G1} and P_{G2} are both at 80 MW, $P_{23} = P_{G1}/3 + 2P_{G2}/3$ also hits the limit. There is no way to generate more than 160 MW from generators 1 and 2 since if we add the two line flow constraints $P_{13} \le 80, P_{23} \le 80$, we get $P_{G1} + P_{G2} \le 160$. The additional power has to be generated by generator 3 so the LMP is \$15. The plot is shown in Fig. 2. ∎

b) Develop the same plot, but considering $N - 1$ transmission line contingencies.

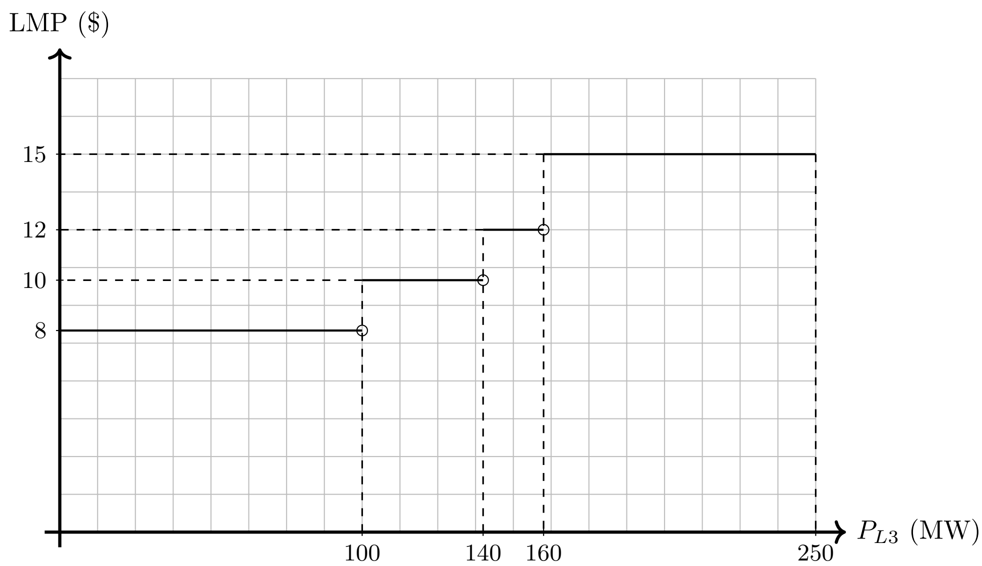

Figure 2: LMP for Problem 2a).

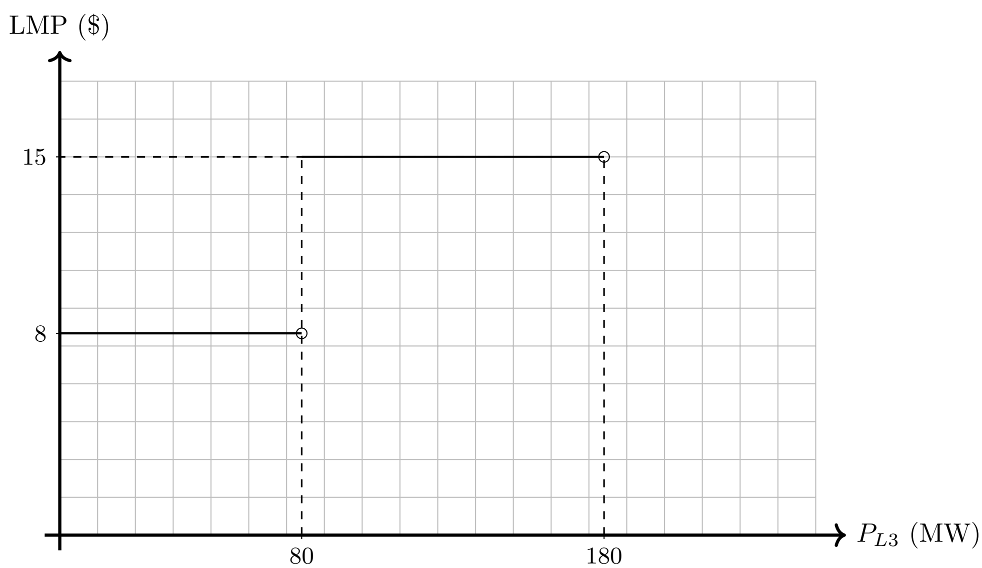

Figure 3: LMP for Problem 2b).

Solution. We need to make sure the generation dispatch is the most economic and satisfies the line flow constraints under all four scenarios (base case and three contingencies). The generation output isn't affected by the scenario, but the line flow is. For each transmission line, we need to identify the scenario under which the line flow is the highest, and ensure the constraint is satisfied by the dispatch under that scenario.

It is not hard to verify that flow on line 1-2 is the highest when either line 1-3 or line 2-3 is outaged. Similarly, flow on line 1-3 (2-3) is the highest when line 2-3 (1-3) is outaged.

When the load is less than 80 MW, the power is supplied entirely from generator 1 and the LMP at bus 3 is \$8. The sum of power output from generators 1 and 2 can't go beyond 80 MW since the line flow on line 1-2 (1-3) is $P_{G1}+P_{G2}$ when line 1-3 (1-2) is outaged. So when load is between 80 and 180 MW, the incremental generation is supplied by generator 3 and the marginal cost is \$15. The load can't be served if it is higher than 180 MW due to the generation constraint at generator 3. The plot of LMP is shown in Fig. 3. ■

Problem 3 (10+10+10+10+10 points)

Suppose we are solving the following standard form OPF problem using Simplex method:

$$\min \quad \sum_{i=1}^{n} C_i(P_{Gi})$$
$$A\begin{bmatrix} P_G \\ x \end{bmatrix} = b, \quad \begin{bmatrix} P_G \\ x \end{bmatrix} \geq 0$$

The basis matrix at the optimal solution is:

$$A_B = \boxed{\begin{array}{cc} P_{G1} & P_{G4} \\ 1 & 1 \\ 0 & -0.1372 \end{array}}$$

where the first row corresponds to the area balance constraint and the second row to a transmission line constraint. It is known that at the optimal solution point, P_{G1} operates at an incremental cost of 12.5 \$/MWh and that P_{G4} operates at an incremental cost of 14 \$/MWh. Determine the LMPs at the buses corresponding to generators 1 and 4 by answering the following questions:

a) For the OPF problem, how many constraints are there in total?

Solution. The number of constraints is the number of rows of A_B, which is <u>two</u>. ■

b) What is the Lagrangian $\mathcal{L}$ for this problem?

Solution. The Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{n} C_i(P_{Gi}) + \lambda^\top \left(A \begin{bmatrix} P_G \\ x \end{bmatrix} - b \right).$$ ■

c) Using the marginal costs of generators 1 and 4, write down the first order necessary conditions for optimality corresponding to $\partial\mathcal{L}/\partial P_{G1}$ and $\partial\mathcal{L}/\partial P_{G4}$.

Solution. The first order conditions are

$$\frac{\partial \mathcal{L}}{\partial P_{G1}} = \frac{\mathrm{d}C_1(P_{G1})}{\mathrm{d}P_{G1}} + (\lambda^\top A)_{P_{G1}} = 0$$
$$\frac{\partial \mathcal{L}}{\partial P_{G4}} = \frac{\mathrm{d}C_4(P_{G4})}{\mathrm{d}P_{G4}} + (\lambda^\top A)_{P_{G4}} = 0$$

where $(\cdot)_{P_{G1}}$ denotes the column of the matrix corresponding to P_{G1}. ■

d) What are the values of the Lagrange multipliers at the optimal solution?

Solution. From the first order condition we have

$$\frac{\mathrm{d}C_1(P_{G1})}{\mathrm{d}P_{G1}} + (\lambda^\top A)_{P_{G1}} = 12.5 + \lambda_1 = 0,$$
$$\frac{\mathrm{d}C_4(P_{G4})}{\mathrm{d}P_{G4}} + (\lambda^\top A)_{P_{G4}} = 14 + \lambda_1 - 0.1372\lambda_4 = 0,$$

which implies

$$\lambda_1 = -12.5, \quad \lambda_4 = 10.93.$$ ■

e) What are the LMPs corresponding to generators 1 and 4?

Solution. The LMPs at buses 1 and 4 are nothing but $-(\lambda^\top A)_{P_{G1}} = -\lambda_1 = 12.5$ and $-(\lambda^\top A)_{P_{G4}} = 0.1372\lambda_4 - \lambda_1 = 14$. ■