EE 553 Homework 2
Solution

Fall 2023

Problem 1 (10+10 points)

Consider the 3-bus system shown in slide 25, Lecture 5.

a) Obtain the PTDF for a transfer from bus 1 to bus 2 using the DC power flow model. Report either the MATLAB code OR the corresponding algebraic solution.

Solution. We know $\mathbf{B}' = \begin{bmatrix} 14 & -10 \\ -10 & 15 \end{bmatrix}$; $\mathbf{T} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. So the angle sensitivities are

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}p} = \left[\mathbf{B}'\right]^{-1}\mathbf{T} = \begin{bmatrix} 0.1364 & 0.0909 \\ 0.0909 & 0.1273 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1364 \\ -0.0909 \end{bmatrix}.$$

The PTDFs can be calculated as the product of negative line susceptance (remember $\mathbf{B}' = -\hat{\mathbf{B}}$ when no shunt is present) and the angle sensitivity difference across the line due to the power transfer. The line susceptance for line 1-2, 1-3, and 2-3 are $b_{12} = -4, b_{13} = -5$, and $b_{23} = -10$, respectively. So the PTDFs under the power transfer $\mathbf{T}$ are

$$\boxed{\mathrm{PTDF}_{12,\mathbf{T}} = -b_{12}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}p} - \frac{\mathrm{d}\theta_2}{\mathrm{d}p}\right) = 4(0 + 0.1364) = 0.5456},$$

$$\mathrm{PTDF}_{13,\mathbf{T}} = -b_{13}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}p} - \frac{\mathrm{d}\theta_3}{\mathrm{d}p}\right) = 5(0 + 0.0909) = 0.4545,$$

$$\mathrm{PTDF}_{23,\mathbf{T}} = -b_{23}\left(\frac{\mathrm{d}\theta_2}{\mathrm{d}p} - \frac{\mathrm{d}\theta_3}{\mathrm{d}p}\right) = 10(-0.1364 + 0.0909) = -0.4550. \qquad \blacksquare$$

b) In the last few months the price of the type of fuel used by generator 2 has increased compared to generator 1. Generator 2 wants to buy power from Generator 1 to meet its obligations with customers. Lines 1-3 and 2-3 have practically infinite thermal limits, but line 1-2 is constrained to transfer 50MW. Using the results in a) of this problem, determine the maximum power transfer capability from bus 1 to 2 while maintaining the same 100MW of load at bus 3.

Solution. The maximum transfer $p_{\max}$ happens when the line flow on line 1-2 is 50MW, i.e.,

$$50 = 25.51 + \mathrm{PTDF}_{12,\mathbf{T}}\,p_{\max} \implies \boxed{p_{\max} = 44.89\mathrm{MW}}. \qquad \blacksquare$$

Problem 2 (20 points)

Figure 1 shows a power system in normal operation (base case). The line parameters are $r = 0, x = 2.0$ with no line charging capacitance, and thermal limit (MVA rating) = 15 MVA. Bus 1 is chosen as the slack bus. Determine the post-contingency flows for an outage of line 1-4 using the DC power flow model. If there is any, report the line overloads assuming the MW limit is equal to the MVA rating of each line.

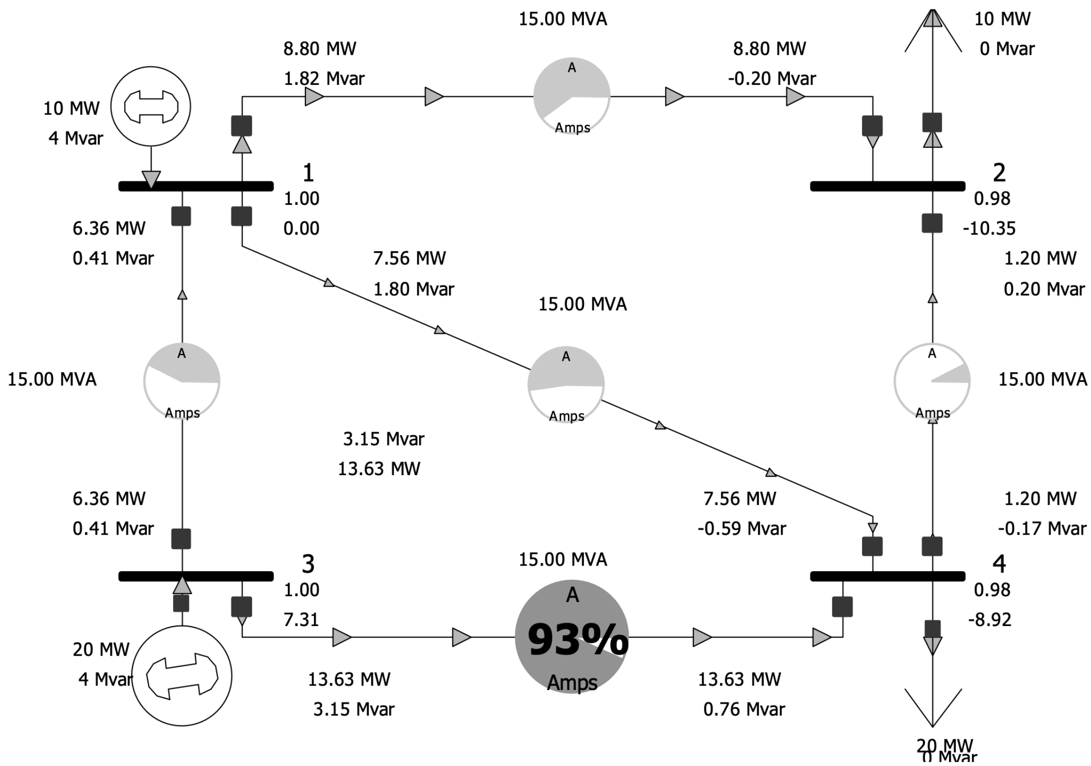


Figure 1: Network for Problem 2.

Solution. We start with building the bus admittance matrix as

$$\mathbf{Y}_{\text{bus}} = \begin{bmatrix} -j1.5 & j0.5 & j0.5 & j0.5 \\ j0.5 & -j1 & 0 & j0.5 \\ j0.5 & 0 & -j1 & j0.5 \\ j0.5 & j0.5 & j0.5 & -j1.5 \end{bmatrix}.$$

Then the $\mathbf{B}'$ matrix, when bus 1 is the slack bus, is

$$\mathbf{B}' = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{bmatrix}.$$

Since we are interested in finding the LODFs for the post-contingency lines, which are

$$\text{LODF}_{12,14} = \frac{\text{PTDF}_{12,\mathbf{T}}}{1 - \text{PTDF}_{14,\mathbf{T}}}, \tag{1a}$$

$$\text{LODF}_{13,14} = \frac{\text{PTDF}_{13,\mathbf{T}}}{1 - \text{PTDF}_{14,\mathbf{T}}}, \tag{1b}$$

$$
\begin{aligned}
\mathrm{LODF}_{24,14} &= \frac{\mathrm{PTDF}_{24,\mathbf{T}}}{1-\mathrm{PTDF}_{14,\mathbf{T}}}, & \text{(1c)}\\
\mathrm{LODF}_{34,14} &= \frac{\mathrm{PTDF}_{34,\mathbf{T}}}{1-\mathrm{PTDF}_{14,\mathbf{T}}}, & \text{(1d)}\\
\mathbf{T} &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, & \text{(1e)}
\end{aligned}
$$

we need to find the PTDFs. To do so, we first find the angle sensitivities as

$$
\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}p} = \left[\mathbf{B}'\right]^{-1}\mathbf{T} = \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \\ -1 \end{bmatrix}.
$$

And the PTDFs are

$$
\begin{aligned}
\mathrm{PTDF}_{12,\mathbf{T}} &= -b_{12}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}p} - \frac{\mathrm{d}\theta_2}{\mathrm{d}p}\right) = 0.5(0+0.5) = 0.25,\\
\mathrm{PTDF}_{13,\mathbf{T}} &= -b_{13}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}p} - \frac{\mathrm{d}\theta_3}{\mathrm{d}p}\right) = 0.5(0+0.5) = 0.25,\\
\mathrm{PTDF}_{14,\mathbf{T}} &= -b_{14}\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}p} - \frac{\mathrm{d}\theta_4}{\mathrm{d}p}\right) = 0.5(0+1) = 0.5,\\
\mathrm{PTDF}_{24,\mathbf{T}} &= -b_{24}\left(\frac{\mathrm{d}\theta_2}{\mathrm{d}p} - \frac{\mathrm{d}\theta_4}{\mathrm{d}p}\right) = 0.5(-0.5+1) = 0.25,\\
\mathrm{PTDF}_{34,\mathbf{T}} &= -b_{34}\left(\frac{\mathrm{d}\theta_3}{\mathrm{d}p} - \frac{\mathrm{d}\theta_4}{\mathrm{d}p}\right) = 0.5(-0.5+1) = 0.25.
\end{aligned}
$$

Plugging back in (1), we have

$$
\begin{aligned}
\mathrm{LODF}_{12,14} &= \frac{\mathrm{PTDF}_{12,\mathbf{T}}}{1-\mathrm{PTDF}_{14,\mathbf{T}}} = 0.5,\\
\mathrm{LODF}_{13,14} &= \frac{\mathrm{PTDF}_{13,\mathbf{T}}}{1-\mathrm{PTDF}_{14,\mathbf{T}}} = 0.5,\\
\mathrm{LODF}_{24,14} &= \frac{\mathrm{PTDF}_{24,\mathbf{T}}}{1-\mathrm{PTDF}_{14,\mathbf{T}}} = 0.5,\\
\mathrm{LODF}_{34,14} &= \frac{\mathrm{PTDF}_{34,\mathbf{T}}}{1-\mathrm{PTDF}_{14,\mathbf{T}}} = 0.5.
\end{aligned}
$$

The post-contingency flows are

$$
\begin{aligned}
&\boxed{P_{12} = P_{12}^0 + \mathrm{LODF}_{12,14}P_{14}^0 = 8.80 + 0.5\times 7.56 = 12.58},\\
&\boxed{P_{13} = P_{13}^0 + \mathrm{LODF}_{13,14}P_{14}^0 = -6.36 + 0.5\times 7.56 = -2.58},\\
&\boxed{P_{24} = P_{24}^0 + \mathrm{LODF}_{24,14}P_{14}^0 = -1.20 + 0.5\times 7.56 = 2.58},\\
&\boxed{P_{34} = P_{34}^0 + \mathrm{LODF}_{34,14}P_{14}^0 = 13.63 + 0.5\times 7.56 = 17.41}.
\end{aligned}
$$

Given line flow limit of 15MW, we see that line 3-4 is overloaded. ■

Problem 3 (20 points)

Solve the following system of linear equations by hand using LU decomposition. Show all steps for 1) the decomposition; and 2) the substitution.

$$\begin{bmatrix} 4 & 2 & 2 & 1 \\ 3 & 6 & 0 & 2 \\ 2 & 2 & 4 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 11 \\ 10 \end{bmatrix} \tag{2}$$

Solution. The general step of using LU decomposition to solve for (2) (which we denote by $\mathbf{Ax} = \mathbf{b}$) is as follows:

1. Obtain the LU decomposition
$$\mathbf{LU} = \mathbf{A}$$
where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is a unit upper triangular matrix (12 pts).
2. Solve for $\mathbf{y} := \mathbf{Ux}$ by substitution using $\mathbf{Ly} = \mathbf{b}$ (6 pts).
3. Solve for $\mathbf{x}$ by substitution using $\mathbf{Ux} = \mathbf{y}$ (2 pts).

Step 1 (LU decomposition):

1. A_{11} (normalization) ($R_1 \leftarrow R_1/4$)
$$\begin{bmatrix} 4 & \color{red}{1/2} & \color{red}{1/2} & \color{red}{1/4} \\ 3 & 6 & 0 & 2 \\ 2 & 2 & 4 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$
2. A_{21} (elimination) ($R_2 \leftarrow R_2 - 3R_1$)
$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & \color{red}{9/2} & \color{red}{-3/2} & \color{red}{5/4} \\ 2 & 2 & 4 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$
3. A_{22} (normalization) ($R_2 \leftarrow 2R_2/9$)
$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & \color{red}{-1/3} & \color{red}{5/18} \\ 2 & 2 & 4 & 3 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$
4. A_{31} (elimination) ($R_3 \leftarrow R_3 - 2R_1$)
$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & -1/3 & 5/18 \\ 2 & \color{red}{1} & \color{red}{3} & \color{red}{5/2} \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

5. A_{32} (elimination) ($R_3 \leftarrow R_3 - R_2$)

$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & -1/3 & 5/18 \\ 2 & 1 & 10/3 & 20/9 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

6. A_{33} (normalization) ($R_3 \leftarrow 3R_3/10$)

$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & -1/3 & 5/18 \\ 2 & 1 & 10/3 & 2/3 \\ 1 & 3 & 2 & 4 \end{bmatrix}$$

7. A_{41} (elimination) ($R_4 \leftarrow R_4 - R_1$)

$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & -1/3 & 5/18 \\ 2 & 1 & 10/3 & 2/3 \\ 1 & 5/2 & 3/2 & 15/4 \end{bmatrix}$$

8. A_{42} (elimination) ($R_4 \leftarrow R_4 - 5R_2/2$)

$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & -1/3 & 5/18 \\ 2 & 1 & 10/3 & 2/3 \\ 1 & 5/2 & 7/3 & 55/18 \end{bmatrix}$$

9. A_{43} (elimination) ($R_4 \leftarrow R_4 - 7R_3/3$)

$$\begin{bmatrix} 4 & 1/2 & 1/2 & 1/4 \\ 3 & 9/2 & -1/3 & 5/18 \\ 2 & 1 & 10/3 & 2/3 \\ 1 & 5/2 & 7/3 & 3/2 \end{bmatrix}$$

So the two triangular matrices are

$$\mathbf{L} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 9/2 & 0 & 0 \\ 2 & 1 & 10/3 & 0 \\ 1 & 5/2 & 7/3 & 3/2 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 0 & 1 & -1/3 & 5/18 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Step 2:

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 3 & 9/2 & 0 & 0 \\ 2 & 1 & 10/3 & 0 \\ 1 & 5/2 & 7/3 & 3/2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 11 \\ 11 \\ 10 \end{bmatrix}$$

We solve for $\mathbf{y}$ by substitution as

- $y_1 = 9/4$

- $3y_1 + 9y_2/2 = 11 \implies y_2 = 17/18$
- $2y_1 + y_2 + 10y_3/3 = 11 \implies y_3 = 5/3$
- $y_1 + 5y_2/2 + 7y_3/3 + 3y_4/2 = 10 \implies y_4 = 1$

Step 3:

$$\begin{bmatrix} 1 & 1/2 & 1/2 & 1/4 \\ 0 & 1 & -1/3 & 5/18 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 17/18 \\ 5/3 \\ 1 \end{bmatrix}$$

We solve for $\mathbf{x}$ by substitution as

- $x_4 = 1$
- $x_3 + 2x_4/3 = 5/3 \implies x_3 = 1$
- $x_2 x_3/3 + 5x_4/18 = 17/18 \implies x_2 = 1$
- $x_1 + x_2/2 + x_3/2 + x_4/4 = 9/4 \implies x_1 = 1$ ■

The solution is $\boxed{\mathbf{x} = [1, 1, 1, 1]^\top}$.

Problem 4 (5+15 points)

Consider the problem shown in slide 26, Lecture 7.

a) Since there is a generator and a load at bus 2, without considering the generator reactive power limit, should bus 2 be modeled as a PV or a PQ bus? Why?

Solution. <u>It should be modeled as a PV bus.</u> Let the power generation of the generator be $P_G + jQ_G$ and the demand of the load be $P_L + jQ_L$. The net power injection at the bus is $(P_G - P_L) + j(Q_G - Q_L)$. Since both P_G and P_L are specified, the bus real power injection is fixed and should be enforced. The bus voltage magnitude is also fixed due to the generator voltage control. Since both the real power injection and the voltage magnitude are fixed, the bus is a PV bus. On the other hand, although Q_L is given, Q_G is unspecified, so the net bus reactive power injection is not determined. ■

b) Reproduce the solution to the power flow problem considering generator reactive power limit. Assuming 100 base MVA so that we can divide all the powers shown in the figure by 100 to get per unit values. Report the code and the program output. Show the values of $Q_{G2}, s, \mathbf{f}$, and $\mathbf{x}$ (reactive power generation at bus 2, the status of generator at bus 2, mismatch vector, and state variables) at each iteration. Please use ℓ_2-norm and tolerance $\epsilon = 1 \times 10^{-5}$ as the stopping criterion.

Solution. The MATLAB code is

```
eps = 1e-5;
Qgmin = 0;
Qgmax = 2;
V2sp = 1.05;

% power mismatch vectors and power flow Jacobian in PQ mode
f_pqmin = @(x) [10*x(2)*sin(x(1)) + 3; 10*x(2)^2 - 10*x(2)*cos(x(1)) + 1.5];
f_pqmax = @(x) [10*x(2)*sin(x(1)) + 3; 10*x(2)^2 - 10*x(2)*cos(x(1)) - 0.5];
J_pq = @(x) [10*x(2)*cos(x(1)), 10 * sin(x(1));
```

```
                 10*x(2)*sin(x(1)), 20*x(2)-10*cos(x(1))];

% power mismatch vector and power flow Jacobian in PV mode
f_pv = @(x) 10.5*sin(x(1)) + 3;
J_pv = @(x) 10.5*cos(x(1));

% reactive power generation at bus 2
Qg2 = @(x) 10*x(2)^2 - 10*x(2)*cos(x(1)) + 1.5;

x = [0; 1.05];
err = eps + 1;
iter = 0;
s = "PV";

formatSpec = "Iteration %i: Qg2 = %6.4f, the status is %s, " + ...
    "theta2 = %7.4f, V2 = %6.4f\n             " + ...
    "The mismatch vector is: [";

while err > eps
    if s == "PV"
        x = x - [J_pv(x) \ f_pv(x); 0];
        if Qg2(x) > Qgmax
            snew = "PQmax";
        elseif QG2(x) < Qgmin
            snew = "PQmin";
        end
        mismatch = f_pv(x);
    elseif s == "PQmin"
        x = x - J_pq(x) \ f_pqmin(x);
        if x(2) < V2sp
            snew = "PV";
        end
        mismatch = f_pqmin(x);
    else % s == "PQmax"
        x = x - J_pq(x) \ f_pqmax(x);
        if x(2) > V2sp
            snew = "PV";
        end
        mismatch = f_pqmax(x);
    end

    err = norm(mismatch);
    iter = iter + 1;

    fprintf(formatSpec, iter, norm(Qg2(x)), s, x(1), x(2));
    if length(mismatch) > 1
        fprintf('%g ', mismatch(1:end-1));
    end
    fprintf('%g]\n', mismatch(end));

    s = snew;
end
```

The program output at each iteration is

```
Iteration 1: Qg2 = 2.4507, the status is PV, theta2 = -0.2857, V2 = 1.0500
             The mismatch vector is: [0.0406501]
Iteration 2: Qg2 = 2.0184, the status is PQmax, theta2 = -0.3020, V2 = 1.0063
             The mismatch vector is: [0.0072097 0.0183977]
Iteration 3: Qg2 = 2.0000, the status is PQmax, theta2 = -0.3034, V2 = 1.0041
             The mismatch vector is: [3.18315e-05 4.6328e-05]
Iteration 4: Qg2 = 2.0000, the status is PQmax, theta2 = -0.3034, V2 = 1.0041
             The mismatch vector is: [3.27808e-10 3.80435e-10]
```

■

Problem 5 (8+7+5 points)

Determine the fills that would occur during elimination in the following matrix:

$$
\begin{bmatrix}
\circ & \circ & \circ & \circ & \circ & \circ & & \circ & & \\
\circ & \circ & \circ & & & & & \circ & & \\
\circ & \circ & \circ & & & & & & \circ & \\
\circ & & & \circ & & & \circ & & & \\
\circ & & & & \circ & & & & & \\
\circ & & & & & \circ & & \circ & \circ & \\
& & & \circ & & & \circ & \circ & & \circ \\
\circ & \circ & & & & \circ & \circ & \circ & \circ & \\
& & \circ & & & \circ & & \circ & \circ & \\
& & & & & & \circ & & & \circ
\end{bmatrix}
$$

a) Without ordering.

Solution. Result after eliminating node 1:

$$
\begin{bmatrix}
\circ & \circ & \circ & \circ & \circ & \circ & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \bullet & \circ & \\
\circ & \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & & \\
\circ & \bullet & \bullet & \bullet & \circ & \bullet & & \bullet & & \\
\circ & \bullet & \bullet & \bullet & \bullet & \circ & & \circ & \circ & \\
& & & \circ & & & \circ & \circ & & \circ \\
\circ & \circ & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \\
& & \circ & & & \circ & & \circ & \circ & \\
& & & & & & \circ & & & \circ
\end{bmatrix}
$$

Result after eliminating node 3:

$$
\begin{bmatrix}
\circ & \circ & \circ & \circ & \circ & \circ & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \bullet & \circ & \\
\circ & \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \\
\circ & \bullet & \bullet & \bullet & \circ & \bullet & & \bullet & \bullet & \\
\circ & \bullet & \bullet & \bullet & \bullet & \circ & & \circ & \circ & \\
& & & \circ & & & \circ & \circ & & \circ \\
\circ & \circ & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \\
& & \circ & \bullet & \bullet & \circ & & \circ & \circ & \\
& & & & & & \circ & & & \circ
\end{bmatrix}
$$

Result after eliminating node 4:

$$
\begin{bmatrix}
\circ & \circ & \circ & \circ & \circ & \circ & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \bullet & \circ & \\
\circ & \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \\
\circ & \bullet & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \\
\circ & \bullet & \bullet & \bullet & \bullet & \circ & \bullet & \circ & \circ & \\
 & & & \circ & \bullet & \bullet & \circ & \circ & \bullet & \circ \\
\circ & \circ & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \\
 & & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \\
 & & & & & & \circ & & & \circ
\end{bmatrix}
$$

Final result (after eliminating node 7):

$$
\boxed{
\begin{bmatrix}
\circ & \circ & \circ & \circ & \circ & \circ & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \circ & & \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & & \bullet & \circ & \\
\circ & \bullet & \bullet & \circ & \bullet & \bullet & \circ & \bullet & \bullet & \\
\circ & \bullet & \bullet & \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \\
\circ & \bullet & \bullet & \bullet & \bullet & \circ & \bullet & \circ & \circ & \\
 & & & \circ & \bullet & \bullet & \circ & \circ & \bullet & \circ \\
\circ & \circ & \bullet & \bullet & \bullet & \circ & \circ & \circ & \circ & \bullet \\
 & & \circ & \bullet & \bullet & \circ & \bullet & \circ & \circ & \bullet \\
 & & & & & & \circ & \bullet & \bullet & \circ
\end{bmatrix}
}
$$

$\blacksquare$

b) Using Tinney 1 ordering method.

Solution. In Tinney 1, the nodes are reordered in ascending order of their degrees. Seen from the matrix, their original degrees and updated node numbers are

Node #	Degree	New Node #
1	6	10
2	3	4
3	3	5
4	2	3
5	1	1
6	3	6
7	3	7
8	5	9
9	3	8
10	1	2

The permuted matrix becomes

$$\begin{bmatrix} \circ & & & & & & & & & \circ \\ & \circ & & & & \circ & & & & \\ & & \circ & & & & \circ & & & \circ \\ & & & \circ & \circ & & & & \circ & \circ \\ & & & \circ & \circ & & & \circ & & \circ \\ & & & & & \circ & & \circ & \circ & \circ \\ & \circ & \circ & & & & \circ & & \circ & \\ & & & & \circ & \circ & & \circ & \circ & \\ & & & \circ & & \circ & \circ & \circ & \circ & \circ \\ \circ & & \circ & \circ & \circ & \circ & & & \circ & \circ \end{bmatrix}$$

Result after eliminating node 3:

$$\begin{bmatrix} \circ & & & & & & & & & \circ \\ & \circ & & & & & \circ & & & \\ & & \circ & & & & \circ & & & \circ \\ & & & \circ & \circ & & & & \circ & \circ \\ & & & \circ & \circ & & & \circ & & \circ \\ & & & & & \circ & & \circ & \circ & \circ \\ & \circ & \circ & & & & \circ & & \circ & \bullet \\ & & & & \circ & \circ & & \circ & \circ & \\ & & & \circ & & \circ & \circ & \circ & \circ & \circ \\ \circ & & \circ & \circ & \circ & \circ & \bullet & & \circ & \circ \end{bmatrix}$$

Result after eliminating node 4:

$$\begin{bmatrix} \circ & & & & & & & & & \circ \\ & \circ & & & & & \circ & & & \\ & & \circ & & & & \circ & & & \circ \\ & & & \circ & \circ & & & & \circ & \circ \\ & & & \circ & \circ & & & \circ & \bullet & \circ \\ & & & & & \circ & & \circ & \circ & \circ \\ & \circ & \circ & & & & \circ & & \circ & \bullet \\ & & & & \circ & \circ & & \circ & \circ & \\ & & & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\ \circ & & \circ & \circ & \circ & \circ & \bullet & & \circ & \circ \end{bmatrix}$$

Final result (after eliminating node 5):

$$\boxed{\begin{bmatrix} \circ & & & & & & & & & \circ \\ & \circ & & & & & \circ & & & \\ & & \circ & & & & \circ & & & \circ \\ & & & \circ & \circ & & & & \circ & \circ \\ & & & \circ & \circ & & & \circ & \bullet & \circ \\ & & & & & \circ & & \circ & \circ & \circ \\ & \circ & \circ & & & & \circ & & \circ & \bullet \\ & & & & \circ & \circ & & \circ & \circ & \bullet \\ & & & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\ \circ & & \circ & \circ & \circ & \circ & \bullet & \bullet & \circ & \circ \end{bmatrix}} \qquad \blacksquare$$

c) Using Tinney 2 ordering method.

Solution. In Tinney 1, the nodes are reordered in ascending order of their degrees at the beginning and after the elimination of every node. We start the process with the same matrix we obtained in Tinney 1:

$$\begin{bmatrix}
\circ & & & & & & & & & \circ \\
& \circ & & & & & \circ & & & \\
& & \circ & & & & \circ & & & \circ \\
& & & \circ & \circ & & & & \circ & \circ \\
& & & \circ & \circ & & & \circ & & \circ \\
& & & & & \circ & & \circ & \circ & \circ \\
& \circ & \circ & & & & \circ & & \circ & \\
& & & & \circ & \circ & & \circ & \circ & \\
& & & \circ & & \circ & \circ & \circ & \circ & \circ \\
\circ & & \circ & \circ & \circ & \circ & & & \circ & \circ
\end{bmatrix}$$

Result after eliminating node 3:

$$\begin{bmatrix}
\circ & & & & & & & & & \circ \\
& \circ & & & & & \circ & & & \\
& & \circ & & & & \circ & & & \circ \\
& & & \circ & \circ & & & & \circ & \circ \\
& & & \circ & \circ & & & \circ & & \circ \\
& & & & & \circ & & \circ & \circ & \circ \\
& \circ & \circ & & & & \circ & & \circ & \bullet \\
& & & & \circ & \circ & & \circ & \circ & \\
& & & \circ & & \circ & \circ & \circ & \circ & \circ \\
\circ & & \circ & \circ & \circ & \circ & \bullet & & \circ & \circ
\end{bmatrix}$$

Upon checking the updated degrees, nodes 7 and 8 needs to be swapped:

Node #	Degree	New Node #
1	1	1
2	1	2
3	2	3
4	3	4
5	3	5
6	3	6
7	4	8
8	3	7
9	5	9
10	7	10

The updated matrix is:

$$\begin{bmatrix}
\circ & & & & & & & & & \circ \\
& \circ & & & & & \circ & & & \\
& & \circ & & & & \circ & & & \circ \\
& & & \circ & \circ & & & & \circ & \circ \\
& & & \circ & \circ & & \circ & & & \circ \\
& & & & & \circ & \circ & & \circ & \circ \\
& & & & \circ & \circ & \circ & & \circ & \\
& \circ & \circ & & & & & \circ & \circ & \bullet \\
& & & \circ & & \circ & \circ & \circ & \circ & \circ \\
\circ & & \circ & \circ & \circ & \circ & & \bullet & \circ & \circ
\end{bmatrix}$$

We continue the elimination of node 4, the resulting matrix is

$$\begin{bmatrix}
\circ & & & & & & & & & \circ \\
& \circ & & & & & \circ & & & \\
& & \circ & & & & \circ & & & \circ \\
& & & \circ & \circ & & & & \circ & \circ \\
& & & \circ & \circ & & \circ & & \bullet & \circ \\
& & & & & \circ & \circ & & \circ & \circ \\
& & & & \circ & \circ & \circ & & \circ & \\
& \circ & \circ & & & & & \circ & \circ & \bullet \\
& & & \circ & \bullet & \circ & \circ & \circ & \circ & \circ \\
\circ & & \circ & \circ & \circ & \circ & & \bullet & \circ & \circ
\end{bmatrix}$$

Upon checking the updated degrees, nodes 5 and 7 needs to be swapped:

Node #	Degree	New Node #
1	1	1
2	1	2
3	2	3
4	3	4
5	4	7
6	3	6
7	3	5
8	4	8
9	6	9
10	7	10

The updated matrix is:

$$\begin{bmatrix}
\circ & & & & & & & & & \circ \\
& \circ & & & \circ & & & & & \\
& & \circ & & \circ & & & & & \circ \\
& & & \circ & & & \circ & & \circ & \circ \\
& & & & \circ & \circ & \circ & & \circ & \\
& & & & \circ & \circ & & & \circ & \circ \\
& & & \circ & \circ & & \circ & & \bullet & \circ \\
& \circ & \circ & & & & & \circ & \circ & \bullet \\
& & & \circ & \circ & \circ & \bullet & \circ & \circ & \circ \\
\circ & & \circ & \circ & & \circ & \circ & \bullet & \circ & \circ
\end{bmatrix}$$

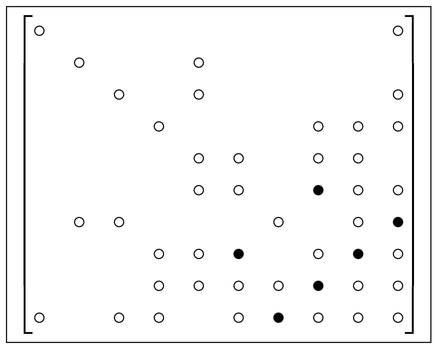
We continue the elimination of node 5, the resulting matrix is

$$\begin{bmatrix}
\circ & & & & & & & & & \circ \\
& \circ & & & \circ & & & & & \\
& & \circ & & \circ & & & & & \circ \\
& & & \circ & & & \circ & & \circ & \circ \\
& & & & \circ & \circ & \circ & & \circ & \\
& & & & \circ & \circ & \bullet & & \circ & \circ \\
& & & \circ & \circ & \bullet & \circ & & \bullet & \circ \\
& \circ & \circ & & & & & \circ & \circ & \bullet \\
& & & \circ & \circ & \circ & \bullet & \circ & \circ & \circ \\
\circ & & \circ & \circ & & \circ & \circ & \bullet & \circ & \circ
\end{bmatrix}$$

Node #	Degree	New Node #
1	1	1
2	1	2
3	2	3
4	3	4
5	3	5
6	4	6
7	5	8
8	4	7
9	6	9
10	7	10

Upon checking the updated degrees, nodes 7 and 8 needs to be swapped:

The updated matrix is:



We continue the elimination. However, no more fills are added. So the above matrix is final. ■

Bonus Problem (10 points)

Suppose we have a function `ludecomp()` at our disposal, which performs LU decomposition and returns the lower triangular matrix $\mathbf{L}$ and the unit upper triangular matrix $\mathbf{U}$. What is the easiest way you can think of to get the alternative LU decomposition with unit lower triangular matrix instead?

Solution. Suppose the matrix we want to factorize is $\mathbf{A}$. We can use the function to perform the LU decomposition of $\mathbf{A}^\top$ to get the lower triangular matrix $\mathbf{L}$ and unit upper triangular matrix $\mathbf{U}$, then $\mathbf{U}' = \mathbf{L}^\top$ is an upper triangular matrix and $\mathbf{L}' = \mathbf{U}^\top$ is a unit lower triangular matrix such that $\mathbf{L}'\mathbf{U}' = \mathbf{A}$. ■