EE 553 Homework 1
Solution

Fall 2023

Problem 1 (10+5+5 points)

a) Build the bus admittance matrix for a system with the following data (values in p.u.):

From bus	To bus	Line resistance	Line reactance	Total line charging susceptance
1	2	0	0.5	0
1	3	0	0.333	0
1	4	0	0.5	0
2	3	0	0.25	0
4	3	0	0.4	0

Solution. For a line with line resistance r, line reactance x, and total line charging susceptance $2b$, the line admittance is

$$y = 1/(r + jx) = \frac{r - jx}{(r + jx)(r - jx)} = \frac{r}{r^2 + x^2} - j\frac{x}{r^2 + x^2},$$

and the shunt admittance on either terminal bus is $y_{\text{sh}} = jb$. Let the line admittance at line (i, j) be y_{ij}, the bus admittance matrix can be represented as

$$\begin{aligned}\mathbf{Y}_{\text{bus}} &= \begin{bmatrix} y_{12} + y_{13} + y_{14} & -y_{12} & -y_{13} & -y_{14} \\ -y_{12} & y_{12} + y_{23} & -y_{23} & 0 \\ -y_{13} & -y_{23} & y_{13} + y_{23} + y_{43} & -y_{43} \\ -y_{14} & 0 & -y_{43} & y_{14} + y_{43} \end{bmatrix} \\ &= \boxed{\begin{bmatrix} -j7 & j2 & j3 & j2 \\ j2 & -j6 & j4 & 0 \\ j3 & j4 & -j9.5 & j2.5 \\ j2 & 0 & j2.5 & -j4.5 \end{bmatrix}}. \end{aligned}$$

■

b) How would the bus admittance matrix change if line 1-2 is upgraded to a line with the following parameters?

From bus	To bus	Line resistance	Line reactance	Total line charging susceptance
1	2	0.2	0.5	0.1

Solution. Denote the line admittance after line upgrade by y'_{12}, and the shunt admittance at both terminals by y_1 and y_2, the bus admittance matrix becomes

$$\mathbf{Y}'_{\text{bus}} = \begin{bmatrix} y'_{12}+y_{13}+y_{14}+y_1 & -y'_{12} & -y_{13} & -y_{14} \\ -y'_{12} & y'_{12}+y_{23}+y_2 & -y_{23} & 0 \\ -y_{13} & -y_{23} & y_{13}+y_{23}+y_{43} & -y_{43} \\ -y_{14} & 0 & -y_{43} & y_{14}+y_{43} \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} 0.6897-j6.6741 & -0.6897+j1.7241 & j3 & j2 \\ -0.6897+j1.7241 & 0.6897-j5.6741 & j4 & 0 \\ j3 & j4 & -j9.5 & j2.5 \\ j2 & 0 & j2.5 & -j4.5 \end{bmatrix}}. \qquad \blacksquare$$

c) How would the bus admittance matrix change with a capacitor with $Q = 1$ p.u. under nominal voltage (1 p.u.) connected to bus 4?

Solution. $S = EI^* = EE^*y^* = V^2y^*$, which means $P = \text{Re}(V^2y^*) = gV^2$ and $Q = \text{Im}(V^2y^*) = -bV^2$. Note that the equation $S = VI^*$ implies that the power (and current) flows from the bus to the ground ($S = EI^* = (E-0)I^*$), so the positive direction of Q is out of the network (as a load). Since capacitors inject reactive power into the network, we have $Q = -1$, which means $b = -Q/V^2 = 1$. So the shunt admittance of the capacitor is $y_{\text{sh},4} = j1$. The bus admittance matrix becomes

$$\mathbf{Y}''_{\text{bus}} = \begin{bmatrix} y_{12}+y_{13}+y_{14} & -y_{12} & -y_{13} & -y_{14} \\ -y_{12} & y_{12}+y_{23} & -y_{23} & 0 \\ -y_{13} & -y_{23} & y_{13}+y_{23}+y_{43} & -y_{43} \\ -y_{14} & 0 & -y_{43} & y_{14}+y_{43}+y_{\text{sh},4} \end{bmatrix}$$
$$= \boxed{\begin{bmatrix} -j7 & j2 & j3 & j2 \\ j2 & -j6 & j4 & 0 \\ j3 & j4 & -j9.5 & j2.5 \\ j2 & 0 & j2.5 & -j3.5 \end{bmatrix}}. \qquad \blacksquare$$

Problem 2 (6+7+7 points)

a) Derive real and reactive power balance equations using the bus admittance matrix $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ with bus voltage phasors represented in rectangular coordinates. In other words, write power balance equations in e and f where $E = Ve^{j\theta}$ is the voltage phasor and $e = \mathrm{Re}(E)$ and $f = \mathrm{Im}(E)$.

Solution. The complex power flow equation for bus i is

$$\begin{aligned}
S_i &= E_i I_i^* \\
&= E_i \sum_{j=1}^{N} Y_{ij}^* E_j^* \\
&= (e_i + jf_i) \sum_{k=1}^{N} (G_{ik} - jB_{ik})(e_k - jf_k) \\
&= (e_i + jf_i) \sum_{k=1}^{N} \left((G_{ik}e_k - B_{ik}f_k) - j\left(G_{ik}f_k + B_{ik}e_k\right)\right) \\
&= \underbrace{\sum_{k=1}^{N} \left(G_{ik}(e_ie_k + f_if_k) + B_{ik}(e_kf_i - e_if_k)\right)}_{P_i} + j \underbrace{\sum_{k=1}^{N} \left(G_{ik}(e_kf_i - e_if_k) - B_{ik}(e_ie_k + f_if_k)\right)}_{Q_i}
\end{aligned}$$

So the power balance equations are

$$\boxed{\sum_{k=1}^{N} \left(G_{ik}(e_ie_k + f_if_k) + B_{ik}(e_kf_i - e_if_k)\right) - P_i = 0}$$

$$\boxed{\sum_{k=1}^{N} \left(G_{ik}(e_kf_i - e_if_k) - B_{ik}(e_ie_k + f_if_k)\right) - Q_i = 0}$$

The power balance equations can also be derived from the polar-coordinate ones

$$\sum_{k=1}^{N} \left(V_iV_kG_{ik}\cos(\theta_i - \theta_k) + V_iV_kB_{ik}\sin(\theta_i - \theta_k)\right) - P_i = 0$$
$$\sum_{k=1}^{N} \left(V_iV_kG_{ik}\sin(\theta_i - \theta_k) - V_iV_kB_{ik}\cos(\theta_i - \theta_k)\right) - Q_i = 0$$

by noticing

$$\begin{aligned}
V_iV_k\cos(\theta_i - \theta_k) &= V_iV_k\cos\theta_i\cos\theta_k + V_iV_k\sin\theta_i\sin\theta_k \\
&= e_ie_k + f_if_k
\end{aligned}$$

and

$$\begin{aligned}
V_iV_k\sin(\theta_i - \theta_k) &= V_iV_k\sin\theta_i\cos\theta_k - V_iV_k\cos\theta_i\sin\theta_k \\
&= e_kf_i - e_if_k
\end{aligned}$$

■

b) For a system with 1 slack bus, m PV buses, and n PQ buses, how many equations are needed to solve the power flow problem in rectangular coordinates and what are they? How many variables are needed and what are they?

Solution. There are $1+m+n$ buses in total, so there are $1+m+n$ e's and $1+m+n$ f's. Out of these variables, only the e and f for the slack bus are known. So there are $\underline{2(m+n)}$ $\underline{\text{variables including } (m+n)\ e\text{'s and } (m+n)\ f\text{'s}}$ that we need to solve for. To solve for them, $2(m+n)$ equations are needed, including $\underline{m+n \text{ real power balance equations for PV and PQ}}$ $\underline{\text{buses, } n \text{ reactive power balance equations for PQ buses, and } m \text{ voltage magnitude equations for}}$ $\underline{\text{PV buses } (V_i^2 = e_i^2 + f_i^2)}$. If we index the buses such that buses $1,\ldots,n$ are PQ buses and buses $n+1,\ldots,n+m$ are PV buses, then the specific equations to be solved for are

$$
\begin{aligned}
f_i^P(\mathbf{e},\mathbf{f}) &= \sum_{k=1}^{1+m+n} \left(G_{ik}(e_ie_k + f_if_k) + B_{ik}(e_kf_i - e_if_k)\right) - P_i = 0, && i = 1,\ldots,m+n \\
f_i^Q(\mathbf{e},\mathbf{f}) &= \sum_{k=1}^{1+m+n} \left(G_{ik}(e_kf_i - e_if_k) - B_{ik}(e_ie_k + f_if_k)\right) - Q_i = 0, && i = 1,\ldots,n \\
f_i^W(\mathbf{e},\mathbf{f}) &= e_i^2 + f_i^2 - V_i^2 = 0, && i = n+1,\ldots,n+m \quad \blacksquare
\end{aligned}
$$

c) Given the power flow equations and variables identified in part b), write down the expressions for the diagonal and off-diagonal entries of the power flow Jacobian (such as $\partial P_i/\partial e_i$).

Solution. The power flow Jacobian is

$$
\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}^P}{\partial \mathbf{e}} & \dfrac{\partial \mathbf{f}^P}{\partial \mathbf{f}} \\ \dfrac{\partial \mathbf{f}^Q_{\mathrm{PQ}}}{\partial \mathbf{e}} & \dfrac{\partial \mathbf{f}^Q_{\mathrm{PQ}}}{\partial \mathbf{f}} \\ \dfrac{\partial \mathbf{f}^W_{\mathrm{PV}}}{\partial \mathbf{e}} & \dfrac{\partial \mathbf{f}^W_{\mathrm{PV}}}{\partial \mathbf{f}} \end{bmatrix} = \begin{bmatrix} & \vdots & & & & & & & & \\ \cdots & \dfrac{\partial f_i^P}{\partial e_i} & \cdots & \dfrac{\partial f_i^P}{\partial e_k} & \cdots & \dfrac{\partial f_i^P}{\partial f_i} & \cdots & \dfrac{\partial f_i^P}{\partial f_k} & \\ & & \ddots & & & & & & \\ & \dfrac{\partial f_k^P}{\partial e_i} & \cdots & \dfrac{\partial f_k^P}{\partial e_k} & \cdots & \dfrac{\partial f_k^P}{\partial f_i} & \cdots & \dfrac{\partial f_k^P}{\partial f_k} & \\ & & & & \ddots & & & & \\ & \dfrac{\partial f_i^Q}{\partial e_i} & \cdots & \dfrac{\partial f_i^Q}{\partial e_k} & \cdots & \dfrac{\partial f_i^Q}{\partial f_i} & \cdots & \dfrac{\partial f_i^Q}{\partial f_k} & \\ & & & & & & \ddots & & \\ & \dfrac{\partial f_k^W}{\partial e_i} & \cdots & \dfrac{\partial f_k^W}{\partial e_k} & \cdots & \dfrac{\partial f_k^W}{\partial f_i} & \cdots & \dfrac{\partial f_k^W}{\partial f_k} & \cdots \\ & & & & & & & \vdots & \end{bmatrix}
$$

$$
= \boxed{\begin{bmatrix} & \vdots & & & & & & & & \\ \cdots & 2G_{ii}e_i + \sum\limits_{k=1,k\neq i}^{1+m+n}(G_{ik}e_k - B_{ik}f_k) & \cdots & G_{ik}e_i + B_{ik}f_i & \cdots & 2G_{ii}f_i + \sum\limits_{k=1,k\neq i}^{1+m+n}(G_{ik}f_k - B_{ik}e_k) & \cdots & G_{ik}f_i - B_{ik}e_i & \\ & & \ddots & & & & & & \\ & G_{ki}e_k + B_{ki}f_k & \cdots & 2G_{kk}e_k + \sum\limits_{i=1,i\neq k}^{1+m+n}(G_{ki}e_i - B_{ki}f_i) & \cdots & G_{ki}f_k - B_{ki}e_k & \cdots & 2G_{kk}f_k + \sum\limits_{i=1,i\neq k}^{1+m+n}(G_{ki}f_i - B_{ki}e_i) & \\ & & & & \ddots & & & & \\ & -2B_{ii}e_i - \sum\limits_{k=1,k\neq i}^{1+m+n}(G_{ik}f_k + B_{ik}e_k) & \cdots & G_{ik}f_i - B_{ik}e_i & \cdots & -2B_{ii}f_i + \sum\limits_{k=1,k\neq i}^{1+m+n}(G_{ik}e_k - B_{ik}f_k) & \cdots & -G_{ik}e_i - B_{ik}f_i & \\ & & & & & & \ddots & & \\ & 0 & \cdots & 2e_k & \cdots & 0 & \cdots & 2f_k & \cdots \\ & & & & & & & \vdots & \end{bmatrix}}
$$

■

Problem 3 (20 points)

Write a computer program to solve the following set of equations using Gauss-Seidel method.

$$x_1 + \sqrt[3]{x_2} - 3 = 0$$
$$x_1^2 + 2x_2 + 5 = 0$$

Report the code and the program output. Please use ℓ_2-norm and tolerance $\epsilon = 2 \times 10^{-4}$ as the stopping criterion, and use $x_1 = 5, x_2 = -18$ as the initial guess.

Solution. We first cast the equations in fixed-point form as

$$x_1 = -\sqrt[3]{x_2} + 3$$
$$x_2 = -\frac{x_1^2 + 5}{2}$$

The following MATLAB code is used to solve the problem:

```
err = inf;
eps = 2e-4;
x1 = 5;
x2 = -18;
iter = 0;
formatSpec = 'Iteration %i: mismatch = %5.4f, x1 = %5.4f, x2 = %5.4f\n';

while err > eps
    % Gauss-Seidel iteration
    x1new = -nthroot(x2, 3) + 3;
    x2new = -(x1new^2 + 5)/2;

    % check the mismatch
    err = norm([x1new - x1, x2new - x2]);
    iter = iter + 1;
    fprintf(formatSpec, iter, err, x1new, x2new)

    % update variables
    x1 = x1new;
    x2 = x2new;
end
```

The output is

```
Iteration 1: mismatch = 0.6879, x1 = 5.6207, x2 = -18.2964
Iteration 2: mismatch = 0.0818, x1 = 5.6350, x2 = -18.3769
Iteration 3: mismatch = 0.0221, x1 = 5.6389, x2 = -18.3986
Iteration 4: mismatch = 0.0060, x1 = 5.6399, x2 = -18.4045
Iteration 5: mismatch = 0.0016, x1 = 5.6402, x2 = -18.4061
Iteration 6: mismatch = 0.0004, x1 = 5.6403, x2 = -18.4065
Iteration 7: mismatch = 0.0001, x1 = 5.6403, x2 = -18.4066
```

■

Problem 4 (20 points)

Reproduce the results of the two-bus system Newton-Raphson power flow shown on slide 39, Lecture 3. Report the code and the program output. Show the Jacobian matrix at each iteration. Please use ℓ_2-norm and tolerance $\epsilon = 2 \times 10^{-4}$ as the stopping criterion.

Solution. The MATLAB code is

```
eps = 2e-4;
x = [0; 1];
f = @(x) [10*x(2)*sin(x(1))+2; -10*x(2)*cos(x(1))+10*x(2)^2+1];
J = @(x) [10*x(2)*cos(x(1)), 10 * sin(x(1));
          10*x(2)*sin(x(1)), -10*cos(x(1))+20*x(2)];
iter = 0;

s1 = "Iteration %i: mismatch = %6.4f, ";
s2 = "J = [%7.4f, %7.4f; %7.4f, %7.4f],\n";
s3 = "theta2 = %7.4f, V2 = %7.4f\n";

while norm(f(x)) >= eps
    % Newton-Raphson iteration
    xnew = x - J(x) \ f(x);

    iter = iter + 1;
    Jac = J(xnew);
    fprintf(s1+s2+"             "+s3, iter, norm(f(xnew)), ...
        Jac(1,1), Jac(1,2), Jac(2,1), Jac(2,2), xnew(1), xnew(2));

    % update variables
    x = xnew;
end
```

The program output including the Jacobian at each iteration is

```
Iteration 1: mismatch = 0.3507, J = [ 8.8206, -1.9867; -1.7880,  8.1993],
             theta2 = -0.2000, V2 =  0.9000
Iteration 2: mismatch = 0.0239, J = [ 8.3538, -2.3123; -1.9855,  7.4441],
             theta2 = -0.2333, V2 =  0.8587
Iteration 3: mismatch = 0.0001, J = [ 8.3169, -2.3380; -1.9999,  7.3850],
             theta2 = -0.2360, V2 =  0.8554
```

∎

Problem 5 (20 points)

Use the fast decoupled power flow for the 5-bus network, show the equations to find the capacitance at bus 5 such that the voltage magnitude at that bus is exactly 1.0. Assume that the capacitance can be modeled as a PV bus for this solution. Why is this true?

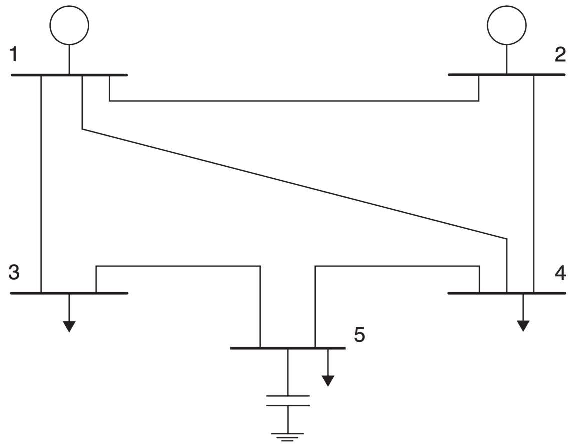

Figure 1: Network for Problem 5.

Table 1: Line data

Y_{12}	Y_{13}	Y_{14}	Y_{24}	Y_{35}	Y_{45}	Y_{S12}
$1-j10$	$1-j10$	$1-j10$	$2-j10$	$1-j5$	$1-j8$	0
Y_{S13}	Y_{S14}	Y_{S24}	Y_{S35}	Y_{S45}	Y_{S54}	Y_{S5}
0	0	0	0	$j0.5$	$j0.5$	$j0.5$

Table 2: Bus data

V_1	δ_1	V_2	P_{G2}	P_{D3}
0.98	0.0	0.97	1.0	0.5
Q_{D3}	P_{D4}	Q_{D4}	P_{D5}	Q_{D5}
0.0	2.0	0.5	0.5	0.1

Solution. The capacitor injects reactive power into the system. To find the exact reactive power injection such as the voltage magnitude is exactly 1.0, the voltage magnitude should be fixed at 1.0 along with the real power when solving the power flow problem. The bus is therefore a PV bus. After obtaining the power flow solutions (step 1), the total reactive power injection $Q_5 = Q_{G5} - Q_{D5}$ can be solved for using Kirchhoff's Current Law (step 2), and the admittance of the capacitor is

$$Y_{\text{cap}} = j\frac{Q_{G5}}{V_5^2} = j\frac{Q_5 + Q_{D5}}{1^2} = j(Q_5 + Q_{D5}). \quad \text{(step 3)}$$

(You only need to show the equations in key steps to receive full credit. It doesn't have to be as complete as this solution.)

Step 1: Solve for the power flow solutions. The bus admittance matrix is

$$\mathbf{Y}_{\text{bus}} = \begin{bmatrix} 3-j30 & -1+j10 & -1+j10 & -1+j10 & 0 \\ -1+j10 & 3-j20 & 0 & -2+j10 & 0 \\ -1+j10 & 0 & 2-j15 & 0 & -1+j5 \\ -1+j10 & -2+j10 & 0 & 4-j27.5 & -1+j8 \\ 0 & 0 & -1+j5 & -1+j8 & 2-j12.5 \end{bmatrix} = \mathbf{Y}_{\text{bus}}^{\text{series}} + \mathbf{Y}_{\text{bus}}^{\text{shunt}}$$

where

$$\mathbf{Y}_{\text{bus}}^{\text{series}} = \begin{bmatrix} 3-j30 & -1+j10 & -1+j10 & -1+j10 & 0 \\ -1+j10 & 3-j20 & 0 & -2+j10 & 0 \\ -1+j10 & 0 & 2-j15 & 0 & -1+j5 \\ -1+j10 & -2+j10 & 0 & 4-j28 & -1+j8 \\ 0 & 0 & -1+j5 & -1+j8 & 2-j13 \end{bmatrix} \quad \text{and}$$

$$\mathbf{Y}_{\text{bus}}^{\text{shunt}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j0.5 & 0 \\ 0 & 0 & 0 & 0 & j0.5 \end{bmatrix}.$$

To apply fast decoupled power flow, we need to build $\mathbf{B}'$ and $\mathbf{B}''$ matrices, which are the submatrices of $-\text{Im}(\mathbf{Y}_{\text{bus}} - \mathbf{Y}_{\text{bus}}^{\text{shunt}})$ and $-\text{Im}(\mathbf{Y}_{\text{bus}} + \mathbf{Y}_{\text{bus}}^{\text{shunt}})$ with relevant rows/columns, respectively:

$$\mathbf{B}' = \begin{bmatrix} 20 & 0 & -10 & 0 \\ 0 & 15 & 0 & -5 \\ -10 & 0 & 28 & -8 \\ 0 & -5 & -8 & 13 \end{bmatrix} \quad \text{and} \quad \mathbf{B}'' = \begin{bmatrix} 15 & 0 \\ 0 & 27 \end{bmatrix}.$$

The power balance equations (where bus 5 is a PV bus) are

$$\begin{aligned}
&2.8227 - 0.9506\cos\theta_2 - 1.94V_4\cos(\theta_2-\theta_4) + 9.506\sin\theta_2 + 9.7V_4\sin(\theta_2-\theta_4) - 1 = 0 \\
&2V_3^2 - 0.98V_3\cos\theta_3 - V_3\cos(\theta_3-\theta_5) + 9.8V_3\sin\theta_3 + 5V_3\sin(\theta_3-\theta_5) + 0.5 = 0 \\
&4V_4^2 - 0.98V_4\cos\theta_4 - 1.94V_4\cos(\theta_4-\theta_2) - V_4\cos(\theta_4-\theta_5) \\
&\qquad + 9.8V_4\sin\theta_4 + 9.7V_4\sin(\theta_4-\theta_2) + 8V_4\sin(\theta_4-\theta_5) + 2 = 0 \\
&2 - V_3\cos(\theta_5-\theta_3) - V_4\cos(\theta_5-\theta_4) + 5V_3\sin(\theta_5-\theta_3) + 8V_4\sin(\theta_5-\theta_4) + 0.5 = 0 \\
&15V_3^2 - 9.8V_3\cos\theta_3 - 5V_3\cos(\theta_3-\theta_5) - 0.98V_3\sin\theta_3 - V_3\sin(\theta_3-\theta_5) = 0 \\
&27.5V_4^2 - 9.8V_4\cos\theta_4 - 9.7V_4\cos(\theta_4-\theta_2) - 8V_4\cos(\theta_4-\theta_5) \\
&\qquad - 0.98V_4\sin\theta_4 - 1.94V_4\sin(\theta_4-\theta_2) - V_4\sin(\theta_4-\theta_5) + 0.5 = 0
\end{aligned}$$

Using fast decoupled power flow, we start from the initial guess of the variables

$$\mathbf{x}^{(0)} = \begin{bmatrix} \theta_2^{(0)} \\ \theta_3^{(0)} \\ \theta_4^{(0)} \\ \theta_5^{(0)} \\ V_3^{(0)} \\ V_4^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

and perform iterative update as

$$\boldsymbol{\theta}^{(v+1)} = \boldsymbol{\theta}^{(v)} - \left[\mathbf{B}'\right]^{-1} \frac{\mathbf{f}^P(\mathbf{x}^{(v)})}{\mathbf{V}^{(v)}}$$

$$\mathbf{V}^{(v+1)} = \mathbf{V}^{(v)} - \left[\mathbf{B}''\right]^{-1} \frac{\mathbf{f}^Q(\mathbf{x}^{(v)})}{\mathbf{V}^{(v)}}$$

where

$$\begin{aligned}
\frac{\mathbf{f}^P(\mathbf{x})}{\mathbf{V}_i} &= \begin{bmatrix} \frac{f_2^P(\mathbf{x})}{V_2} & \frac{f_3^P(\mathbf{x})}{V_3} & \frac{f_4^P(\mathbf{x})}{V_4} & \frac{f_5^P(\mathbf{x})}{V_5} \end{bmatrix}^\top \\
&= \begin{bmatrix} 2.91 - 0.98\cos\theta_2 - 2V_4\cos\theta_{24} + 9.8\sin\theta_2 + 10V_4\sin\theta_{24} - \frac{1}{V_2} \\ 2V_3 - 0.98\cos\theta_3 - \cos\theta_{35} + 9.8\sin\theta_3 + 5\sin\theta_{35} + \frac{0.5}{V_3} \\ 4V_4 - 0.98\cos\theta_4 - 1.94\cos\theta_{42} - \cos\theta_{45} + 9.8\sin\theta_4 + 9.7\sin\theta_{42} + 8\sin\theta_{45} + \frac{2}{V_4} \\ 2 - V_3\cos\theta_{53} - V_4\cos\theta_{54} + 5V_3\sin\theta_{53} + 8V_4\sin\theta_{54} + 0.5 \end{bmatrix} \\
\frac{\mathbf{f}^Q(\mathbf{x})}{\mathbf{V}_i} &= \begin{bmatrix} \frac{f_3^Q(\mathbf{x})}{V_3} & \frac{f_4^Q(\mathbf{x})}{V_4} \end{bmatrix}^\top \\
&= \begin{bmatrix} 15V_3 - 9.8\cos\theta_3 - 5\cos\theta_{35} - 0.98\sin\theta_3 - \sin\theta_{35} \\ 27.5V_4 - 9.8\cos\theta_4 - 9.7\cos\theta_{42} - 8\cos\theta_{45} - 0.98\sin\theta_4 - 1.94\sin\theta_{42} - \sin\theta_{45} + \frac{0.5}{V_4} \end{bmatrix}
\end{aligned}$$

until $\|\mathbf{x}^{(v+1)} - \mathbf{x}^{(v)}\| \le \epsilon$ for some pre-specified $\epsilon > 0$.

The MATLAB code to carry out the fast decoupled power flow is

```
eps = 1e-8;
x = [0; 0; 0; 0; 1; 1];
xnew = zeros(6,1);
fpv = @(x) [2.91 - 0.98*cos(x(1)) - 2*x(6)*cos(x(1)-x(3)) + 9.8*sin(x(1)) ...
                + 10*x(6)*sin(x(1)-x(3)) - 1/0.97;
            2*x(5) - 0.98*cos(x(2)) - cos(x(2)-x(4)) + 9.8*sin(x(2)) ...
                + 5*sin(x(2)-x(4)) + 0.5/x(5);
            4*x(6) - 0.98*cos(x(3)) - 1.94*cos(x(3)-x(1)) - cos(x(3)-x(4)) ...
                + 9.8*sin(x(3)) + 9.7*sin(x(3)-x(1)) + 8*sin(x(3)-x(4)) + 2/x(6);
            2 - x(5)*cos(x(4)-x(2)) - x(6) * cos(x(4)-x(3)) ...
                + 5*x(5)*sin(x(4)-x(2)) + 8*x(6)*sin(x(4)-x(3)) + 0.5];
fqv = @(x) [15*x(5) - 9.8*cos(x(2)) - 5*cos(x(2)-x(4)) - 0.98*sin(x(2)) ...
                - sin(x(2)-x(4));
            27.5*x(6) - 9.8*cos(x(3)) - 9.7*cos(x(3)-x(1)) - 8*cos(x(3)-x(4)) ...
                - 0.98*sin(x(3)) - 1.94*sin(x(3)-x(1)) - sin(x(3)-x(4)) ...
                + 0.5/x(6)];

Ybus = [3-1i*30,    -1+1i*10,   -1+1i*10,   -1+1i*10,   0;
         -1+1i*10,  3-1i*20,    0,          -2+1i*10,   0;
         -1+1i*10,  0,          2-1i*15,    0,          -1+1i*5;
         -1+1i*10,  -2+1i*10,   0,          4-1i*27.5,  -1+1i*8;
         0,         0,          -1+1i*5,    -1+1i*8,    2-1i*12.5];

Yshunt = diag([0; 0; 0; 0.5i; 0.5i]);

Bpinv  = inv(-imag(Ybus(2:5,2:5) - Yshunt(2:5,2:5)));
Bppinv = inv(-imag(Ybus(3:4,3:4) + Yshunt(3:4,3:4)));

```

```
    iter = 0;

    s1 = "Iteration %i: mismatch = %10.8f, V3 = %7.4f, V4 = %7.4f\n";
    s2 = "theta2 = %7.4f, theta3 = %7.4f, theta4 = %7.4f, theta5 = %7.4f\n";

    while norm([fpv(x);fqv(x)]) > eps
        % Newton-Raphson iteration
        xnew = x - [Bpinv  * fpv(x); Bppinv * fqv(x)];

        iter = iter + 1;
        fprintf(s1+'             '+s3, iter, norm([fpv(xnew);fqv(xnew)]), ...
            xnew(5), xnew(6), xnew(1), xnew(2), xnew(3), xnew(4));

        % update variables
        x = xnew;
    end
```

The program output is

```
    Iteration 1: mismatch = 0.44513213, V3 =  0.9867, V4 =  0.9815
             theta2 = -0.0027, theta3 = -0.0817, theta4 = -0.1155, theta5 = -0.1410
    Iteration 2: mismatch = 0.04321240, V3 =  0.9825, V4 =  0.9653
             theta2 = -0.0066, theta3 = -0.0842, theta4 = -0.1206, theta5 = -0.1483
    Iteration 3: mismatch = 0.00513341, V3 =  0.9824, V4 =  0.9649
             theta2 = -0.0072, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1503
    Iteration 4: mismatch = 0.00043555, V3 =  0.9825, V4 =  0.9650
             theta2 = -0.0073, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1505
    Iteration 5: mismatch = 0.00006401, V3 =  0.9825, V4 =  0.9650
             theta2 = -0.0073, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1505
    Iteration 6: mismatch = 0.00000643, V3 =  0.9825, V4 =  0.9650
             theta2 = -0.0073, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1505
    Iteration 7: mismatch = 0.00000075, V3 =  0.9825, V4 =  0.9650
             theta2 = -0.0073, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1505
    Iteration 8: mismatch = 0.00000008, V3 =  0.9825, V4 =  0.9650
             theta2 = -0.0073, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1505
    Iteration 9: mismatch = 0.00000001, V3 =  0.9825, V4 =  0.9650
             theta2 = -0.0073, theta3 = -0.0846, theta4 = -0.1204, theta5 = -0.1505
```

Step 2: Once we have the power flow solution of voltage in terms of magnitudes and phase angles, we can find the complex voltages at all buses. They are

$$\begin{aligned}\mathbf{E} &= [E_1, \ldots, E_5]^\top \\ &= \begin{bmatrix} 0.9800 - j0.0000 \\ 0.9700 - j0.0070 \\ 0.9790 - j0.0831 \\ 0.9580 - j0.1159 \\ 0.9887 - j0.1499 \end{bmatrix}\end{aligned}$$

The reactive power injection at bus 5 is

$$Q_5 = \mathrm{Im}\left(E_5(\mathbf{Y}_{\text{bus}}\mathbf{E})_5^*\right) = -0.0249$$

Step 3: The admittance of the capacitor is

$$Y_{\text{cap}} = j(Q_5 + Q_{D5}) = j(-0.0249 + 0.1) = j0.0751. \quad \blacksquare$$